# MATH 303 – Measures and Integration
# Homework 1

**Problem 1.** Let $U \subseteq \mathbb{R}^d$ be an open set. Show that $U$ can be written as a disjoint union of countably many half-open boxes (i.e., sets of the form $B = \prod_{i=1}^{d}[a_i, b_i)$).

**Solution:** For each $\boldsymbol{n} \in \mathbb{Z}^d$ and $m \geq 0$, let $B(m, \boldsymbol{n}) = 2^{-m}(\boldsymbol{n} + [0,1)^d)$. Note that for each $m$, the family $(B(m,\boldsymbol{n}))_{\boldsymbol{n}\in\mathbb{Z}^d}$ partitions $\mathbb{R}^d$ into disjoint half-open cubes of side length $2^{-m}$.

Let $\boldsymbol{x} \in U$. Then there exists $m \geq 0$ and $\boldsymbol{n} \in \mathbb{Z}^d$ such that $\boldsymbol{x} \in B(m,\boldsymbol{n})$ and $B(m,\boldsymbol{n}) \subseteq U$. Let $m(\boldsymbol{x}) = \min\{m \geq 0 : \exists \boldsymbol{n} \in \mathbb{Z}^d, \boldsymbol{x} \in B(m,\boldsymbol{n}) \subseteq U\}$. Then let $\boldsymbol{n}(\boldsymbol{x}) \in \mathbb{Z}^d$ such that $\boldsymbol{x} \in B(m(\boldsymbol{x}), \boldsymbol{n}(\boldsymbol{x}))$. As noted above, the family $(B(m(\boldsymbol{x}),\boldsymbol{n}))_{\boldsymbol{n}\in\mathbb{Z}^d}$ partitions $\mathbb{R}^d$ into disjoint half-open cubes, so $\boldsymbol{x}$ belongs to exactly one of the cubes, making $\boldsymbol{n}(\boldsymbol{x})$ well-defined. In words, $B(m(\boldsymbol{x}), \boldsymbol{n}(\boldsymbol{x}))$ is the largest "dyadic cube" containing $\boldsymbol{x}$ and fully contained in $U$.

We claim that for $\boldsymbol{x}, \boldsymbol{y} \in U$, either $B(m(\boldsymbol{x}), \boldsymbol{n}(\boldsymbol{x})) \cap B(m(\boldsymbol{y}), \boldsymbol{n}(\boldsymbol{y})) = \emptyset$ or $B(m(\boldsymbol{x}), \boldsymbol{n}(\boldsymbol{x})) = B(m(\boldsymbol{y}), \boldsymbol{n}(\boldsymbol{y}))$. Indeed, if the two sets intersect, then one must be contained in the other. But from the minimality of $m(\boldsymbol{x})$ and $m(\boldsymbol{y})$, this implies that the half-open boxes coincide.

Letting $I = \{(m(\boldsymbol{x}), \boldsymbol{n}(\boldsymbol{x})) : \boldsymbol{x} \in U\}$, we therefore have $U = \bigcup_{(m,\boldsymbol{n})\in I} B(m,\boldsymbol{n})$, which expresses $U$ as a countable disjoint union of half-open boxes.